Large Degree Asymptotics of Generalized Bernoulli and Euler Polynomials

José Luis López
Departamento de Matématica e Informática,
Universidad Pública de Navarra, 31006 Pamplona, Spain

Nico M. Temme
CWI, Science Park 123, 1098 GX Amsterdam, The Netherlands.
e-mail: jl.lopez@unavarra.es, Nico.Temme@cwi.nl

August 21, 2009

Abstract

Asymptotic expansions are given for large values of n of the generalized Bernoulli polynomials $B_n^\mu(z)$ and Euler polynomials $E_n^\mu(z)$. In a previous paper López and Temme (1999) these polynomials have been considered for large values of μ, with n fixed. In the literature no complete description of the large n asymptotics of the considered polynomials is available. We give the general expansions, summarize known results of special cases and give more details about these results. We use two-point Taylor expansions for obtaining new type of expansions. The analysis is based on contour integrals that follow from the generating functions of the polynomials.

2000 Mathematics Subject Classification: 11B68, 30E10, 33E20, 41A60.
Keywords & Phrases: asymptotic expansions, generalized Bernoulli polynomials, generalized Euler polynomials.

1 Introduction

Generalized Bernoulli and Euler polynomials of degree n, complex order μ and complex argument z, denoted respectively by $B_n^\mu(z)$ and $E_n^\mu(z)$, can be defined by their generating functions. We have [5, 10]

$$\frac{w^\mu e^{wz}}{(e^w-1)^\mu} = \sum_{n=0}^{\infty} \frac{B_n^\mu(z)}{n!} w^n, \quad |w| < 2\pi, \tag{1.1}$$

and

$$\frac{2^\mu e^{wz}}{(e^w+1)^\mu} = \sum_{n=0}^{\infty} \frac{E_n^\mu(z)}{n!} w^n, \quad |w| < \pi. \tag{1.2}$$

These polynomials play an important role in the calculus of finite differences. In fact, the coefficients in all the usual central-difference formulae for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of these polynomials (see [5, 7]).

An explicit formula for the generalized Bernoulli polynomials can be found in [9]. Properties and explicit formulas for the generalized Bernoulli and Euler numbers can be found in [4, 11, 12] and in cited references.

In a previous paper [1] we have considered these polynomials for large values of μ, with n fixed, and in the present paper we consider n as the large parameter, with the other parameters fixed. We summarize known results from the literature for integer values of μ, and give more details about these results. We describe the method for obtaining the coefficients in the expansion for general μ. Finally, we use two-point Taylor expansions for obtaining new type of expansions for general μ. The analysis is based on contour integrals that follow from the generating functions of the polynomials.

2 The generalized Bernoulli polynomials

Three different cases arise, depending on $\mu = 0, -1, -2, \ldots$, $\mu = 1, 2, 3, \ldots$, and μ otherwise, real or complex. Our approach is based on the Cauchy integral

$$B_n^\mu(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{w^\mu e^{wz}}{(e^w-1)^\mu} \frac{dw}{w^{n+1}}, \tag{2.1}$$

where $\mathcal{C}$ is a circle around the origin, with radius less than 2π. This follows from (1.1).

2.1 Asymptotic form when $\mu = 0, -1, -2, \ldots$

In this case the generating series in (1.1) converges for all finite values of z, and, hence, the polynomials $B_n^\mu(z)$ have a completely different behavior compared with the general case.

We first follow the approach given in [14], and observe that when μ is a negative integer or zero, say $\mu = -m$ $(m = 0, 1, 2, \ldots)$, we can express $B_n^\mu(z)$ in terms of a finite sum. We expand by the binomial theorem

$$(e^w - 1)^m = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} e^{rw}. \tag{2.2}$$

This gives

$$B_n^{-m}(z) = \frac{n!}{(n+m)!} \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} (z+r)^{n+m}. \tag{2.3}$$

For any given $m \in \mathbb{N}$ and complex z only the term or terms with the largest values of $|z+r|$ will give a large contribution to the sum in (2.3), the other terms being exponentially small in comparison. We conclude, that (2.3) gives the asymptotic form when $n \to \infty$, when $\mu = -m$ and z are fixed.

In particular, when $z > 0$, the term with index $r = m$ is maximal, and we have

$$B_n^{-m}(z) = \frac{n!}{(n+m)!}(z+m)^{n+m} \left[1 + \mathcal{O}\left(\frac{z+m-1}{z+m}\right)^{n+m}\right]. \tag{2.4}$$

The error term can also be estimated by $\mathcal{O}(\exp(-(n+m)/(z+m)))$, which is indeed exponentially small compared with unity.

For general complex $z = x + iy$ and $x > -m/2$ the term with index $r = m$ again is maximal and the same estimate as in (2.4) is valid. When $x = -m/2$ the terms with $r = 0$ and $r = m$ give the maximal contributions, and we have

$$B_n^{-m}(z) \sim \frac{n!}{(n+m)!} \left[(-1)^m(-\tfrac{1}{2}m + iy)^{n+m} + (\tfrac{1}{2}m + iy)^{n+m}\right]. \tag{2.5}$$

When $x < -m/2$ the term with index $r = 0$ is maximal, and we have

$$B_n^{-m}(z) = (-1)^m \frac{n!}{(n+m)!} z^{n+m} \left[1 + \mathcal{O}\left(\frac{z+1}{z}\right)^{n+m}\right]. \tag{2.6}$$

By using the saddle point method we can obtain an estimate similar as the one in (2.4). We write

$$B_n^{-m}(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \left(1 - e^{-w}\right)^m e^{\phi(w)} \frac{dw}{w}, \tag{2.7}$$

where $\phi(w) = (z+m)w - (n+m)\ln w$. This function has a saddle point at $w_0 = (n+m)/(z+m)$, and when $\Re w_0$ is large and positive we replace $(1-e^{-w})$ by its value at this point. Then we have

$$\begin{aligned} B_n^{-m}(z) &\sim \left(1-e^{-w_0}\right)^m \frac{n!}{2\pi i}\int_{\mathcal{C}} e^{(z+m)w}\frac{dw}{w^{n+m+1}} \\ &= \left(1-e^{-w_0}\right)^m \frac{n!}{(n+m)!}(z+m)^{n+m}, \end{aligned} \tag{2.8}$$

in which e^{-w_0} is exponentially small. When $\Re w_0$ is not positive we can modify this method to obtain the estimates as given in (2.5) and (2.6). Also, we can make further steps in the saddle point analysis, and show that the next terms in the expansion are exponentially small compared with unity, similar as shown in (2.4). However, the representation in (2.3) describes very elegantly the asymptotic behavior.

2.2 Asymptotic form when $\mu = 1, 2, 3, \ldots$

We write $\mu = m$. The starting point is the expansion for $m = 1$:

$$B_n^1(z) = B_n(z) = -n! \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k z}}{(2\pi i k)^n}, \tag{2.9}$$

which for $z \in (0,1)$ can be viewed as a Fourier expansion of $B_n^1(z)$. This expansion follows from taking the radius of the circle $\mathcal{C}$ in (2.1) equal to $(2K+1)\pi$, (K an integer). Taking K large, we take into account the poles of the integrand at $w = 2\pi i k$ ($k = \pm 1, \pm 2, \ldots$), and calculate the residues of these poles. The integral around the circle $\mathcal{C}$ tends to zero as $K \to \infty$, provided $n > 1$ and $0 \le z \le 1$.

This gives the expansions ($n = 1, 2, 3, \ldots; 0 \le z \le 1$)

$$B_{2n}(z) = 2(-1)^{n+1}(2n)! \sum_{k=1}^{\infty} \frac{\cos(2\pi k z)}{(2\pi k)^{2n}}, \tag{2.10}$$

and

$$B_{2n+1}(z) = 2(-1)^{n+1}(2n+1)! \sum_{k=1}^{\infty} \frac{\sin(2\pi k z)}{(2\pi k)^{2n+1}}. \tag{2.11}$$

In (2.11) we can take $n = 0$, provided $0 < z < 1$. This gives the well-known Fourier expansion of $B_1(z) = z - 1/2$, $0 < z < 1$.

In (2.9) only the terms with $k = \pm 1$ are relevant for the asymptotic behavior, and we obtain for fixed complex z

$$B_n^1(z) = \frac{2(-1)^{n+1} n!}{(2\pi)^n} \left[\cos\left(2\pi z + \tfrac{1}{2}\pi n\right) + \mathcal{O}\left(2^{-n}\right)\right], \quad n \to \infty. \tag{2.12}$$

For general fixed real or complex z the series in (2.10) and (2.11) can be viewed as asymptotic expansion for large n, as easily follows from the ratio test.

For general $\mu = m = 1, 2, 3, \ldots$ a similar expansion as in (2.9) can be given. In that case the poles of the integrand in (2.1) are of higher order. We can write

$$B_n^m(z) = -n! \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \beta_k^m(n, z) \frac{e^{2\pi i k z}}{(2\pi i k)^n}, \tag{2.13}$$

where $\beta_k^1(n, z) = 1, \forall k$. This is a Fourier expansion for $z \in (0, 1)$ when $m < n$. For other values of z it can be used as an asymptotic expansion for large n.

An explicit form of $\beta_k^m(n, z)$ follows from calculating the residues of the poles at $2\pi i k$ of order m of the integrand in (2.1). For this we compute the coefficient c_{m-1} in the expansion

$$\frac{(w - 2\pi i k)^m w^m e^{zw}}{(e^w - 1)^m w^{n+1}} = \sum_{r=0}^{\infty} c_r (w - 2\pi i k)^r, \tag{2.14}$$

from which we obtain

$$\beta_k^m(n, z) \frac{e^{2\pi i k z}}{(2\pi i k)^n} = c_{m-1}. \tag{2.15}$$

We substitute $w = s + 2\pi i k$ and write the expansion as

$$e^{2\pi i k z} \frac{s^m e^{zs}}{(e^s - 1)^m (s + 2\pi i k)^{n+1-m}} = \sum_{r=0}^{\infty} c_r s^r. \tag{2.16}$$

We use (1.1) and write the left-hand side in the form

$$\frac{e^{2\pi i k z}}{(2\pi i k)^{n+1-m}} \sum_{\nu=0}^{\infty} \frac{B_\nu^m(z)}{n!} s^\nu \sum_{\nu=0}^{\infty} \binom{m-n-1}{\nu} \frac{s^\nu}{(2\pi i k)^\nu}. \tag{2.17}$$

Hence, c_r of (2.16) can be written as

$$c_r = \frac{e^{2\pi i k z}}{(2\pi i k)^{n+1-m}} \sum_{\nu=0}^{r} \frac{B_\nu^m(z)}{\nu!} \binom{m-n-1}{r-\nu} (2\pi i k)^{\nu - r}, \tag{2.18}$$

and we conclude that

$$c_{m-1} = \frac{e^{2\pi ikz}}{(2\pi ik)^n} \sum_{\nu=0}^{m-1} \frac{B_\nu^m(z)}{\nu!} \binom{m-n-1}{m-1-\nu} (2\pi ik)^\nu. \tag{2.19}$$

It follows from (2.15) that

$$\beta_k^m(n,z) = \sum_{\nu=0}^{m-1} \frac{B_\nu^m(z)}{\nu!} \binom{m-n-1}{m-1-\nu} (2\pi ik)^\nu. \tag{2.20}$$

To avoid binomials with negative integers, and to extract the main asymptotic factor, we write

$$\beta_k^m(n,z) = (-1)^{m-1} \binom{n-1}{m-1} \sum_{\nu=0}^{m-1} B_\nu^m(z) \binom{m-1}{\nu} \frac{(n-\nu-1)!}{(n-1)!} (-2\pi ik)^\nu. \tag{2.21}$$

For large n the main term occurs for $\nu = 0$. We have

$$\beta_k^m(n,z) = \frac{(-1)^{m-1} n^{m-1}}{(m-1)!} \left[1 + \mathcal{O}(n^{-1})\right]. \tag{2.22}$$

Observing that, as in (2.9), only the terms with $k = \pm 1$ are relevant for the asymptotic behavior, we obtain

$$B_n^m(z) = \frac{(-1)^{n+1} n!}{(2\pi)^n} \left[\beta_1^m(n,z) e^{2\pi iz + \frac{1}{2}\pi in} + \beta_{-1}^m(n,z) e^{-2\pi iz - \frac{1}{2}\pi in} + \ldots\right], \tag{2.23}$$

and by using (2.21) we obtain for fixed m and complex z (cf. (2.12))

$$\begin{aligned} B_n^m(z) = {} & \frac{2(-1)^{m+n}}{(2\pi)^n} \binom{n-1}{m-1} \times \\ & \left[\sum_{\nu=0}^{m-1} B_\nu^m(z) \binom{m-1}{\nu} \frac{(n-\nu-1)!}{(n-1)!} (2\pi)^\nu \cos\sigma + \mathcal{O}\left(2^{-n}\right)\right], \end{aligned} \tag{2.24}$$

as $n \to \infty$, where $\sigma = (2z + \frac{1}{2}n - \frac{1}{2}\nu)\pi$.

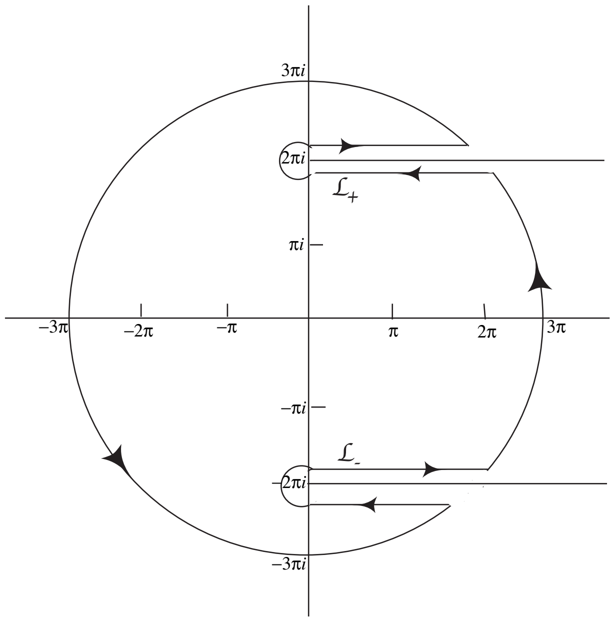
To obtain $\beta_k^m(n,z)$ for $m > 1$ we can also use a recurrence relation. We have the relation

$$\mu B_n^{\mu+1}(z) = (\mu - n) B_n^\mu(z) + n(z-\mu) B_{n-1}^\mu(z), \quad n \geq 1, \tag{2.25}$$

which follows from (1.1) by differentiating both members with respect to w. By differentiation with respect to z we find

$$n B_{n-1}^\mu(z) = \frac{d}{dz} B_n^\mu(z), \tag{2.26}$$

Figure 1: Contour for (2.1) for general μ



giving

$$\mu B_n^{\mu+1}(z) = (\mu - n)B_n^{\mu}(z) + (z-\mu)\frac{d}{dz}B_n^{\mu}(z), \quad n \ge 0. \tag{2.27}$$

This gives the recurrence relation for $m = 1, 2, 3, \dots$

$$m\beta_k^{m+1}(n,z) = [m-n+2\pi i k(z-m)]\beta_k^m(n,z) + (z-m)\frac{d}{dz}\beta_k^m(n,z). \tag{2.28}$$

2.3 Asymptotic form for general complex μ

We consider (2.1) and observe that the singularities at $\pm 2\pi i$ are the sources for the main asymptotic contributions. We integrate around a circle with radius 3π, avoiding branch cuts running from $\pm 2\pi i$ to $+\infty$. See Figure 1. The contribution from the circular arc is $\mathcal{O}((3\pi)^{-n})$, which is exponentially small with respect to the main contributions.

We denote the loops by $\mathcal{L}_\pm$ and the contributions from the loops by $I_\pm$. For the upper loop we substitute $w = 2\pi i e^s$. This gives

$$I_+ = \frac{n!}{2\pi i}\frac{e^{2\pi i z}}{(2\pi i)^n}\int_{\mathcal{C}_+} g(s)s^{-\mu}e^{-ns}\,ds, \tag{2.29}$$

where

$$g(s) = \left(\frac{2\pi i s}{e^u - 1}\right)^{\mu} e^{zu+\mu s}, \quad u = 2\pi i\,(e^s - 1)\,, \tag{2.30}$$

and $\mathcal{C}_+$ is the image of $\mathcal{L}_+$. $\mathcal{C}_+$ is a contour that encircles the origin in the clockwise fashion.

To obtain an asymptotic expansion we apply Watson's lemma for loop integrals, see [8, p. 120]. We expand

$$g(s) = \sum_{k=0}^{\infty} g_k s^k, \tag{2.31}$$

substitute this in (2.29), and interchange summation and integration. This gives

$$I_+ \sim n!\,\frac{e^{2\pi i z}}{(2\pi i)^n} \sum_{k=0}^{\infty} g_k F_k, \tag{2.32}$$

where

$$F_k = \frac{1}{2\pi i}\int_{\mathcal{C}_+} s^{k-\mu} e^{-ns}\,ds, \tag{2.33}$$

with $\mathcal{C}_+$ extended to $+\infty$. That is, we start the integration along the contour $\mathcal{C}_+$ at $s = +\infty$, with $\operatorname{ph} s = 2\pi$, turn around the origin in the clock-wise direction, and return to $+\infty$ with $\operatorname{ph} s = 0$.

To evaluate the integrals we turn the path by writing $s = e^{\pi i}t$, and use the representation of the reciprocal gamma function in terms of the Hankel contour; see [10, p. 48]. The result is

$$F_k = n^{\mu-k-1} e^{\pi i \mu} \frac{(-1)^k}{\Gamma(\mu - k)} = n^{\mu-k-1} e^{\pi i \mu} \frac{(1-\mu)_k}{\Gamma(\mu)}, \tag{2.34}$$

where $(a)_n$ is the shifted factorial, or Pochhammer's symbol, defined by

$$(a)_n = a\cdot(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 0,1,2,\ldots. \tag{2.35}$$

This gives the expansion

$$I_+ \sim \frac{n!\,n^{\mu-1}}{(2\pi)^n\,\Gamma(\mu)}\, e^{i\chi} \sum_{k=0}^{\infty} \frac{(1-\mu)_k g_k}{n^k}, \tag{2.36}$$

where

$$\chi = 2\zeta - \tfrac{1}{2}n\pi, \quad \zeta = (z + \tfrac{1}{2}\mu)\pi. \tag{2.37}$$

The contribution I_- can be obtained in a similar way. However, it is the complex conjugate of I_+ (not considering z and μ as complex numbers). The result $I_+ + I_-$ can be obtained by taking twice the real part of I_+. We write $g_k = g_k^{(r)} + i g_k^{(i)}$ (with $g_k^{(r)}, g_k^{(i)}$ real when z and μ are real), and obtain

$$B_n^\mu(z) \sim \frac{2\, n!\, n^{\mu-1}}{(2\pi)^n\, \Gamma(\mu)} \left[\cos\chi \sum_{k=0}^{\infty} \frac{(1-\mu)_k g_k^{(r)}}{n^k} - \sin\chi \sum_{k=0}^{\infty} \frac{(1-\mu)_k g_k^{(i)}}{n^k}\right], \tag{2.38}$$

as $n \to \infty$, with z and μ fixed complex numbers ($\mu \notin \mathbb{Z}$).

The first few coefficients $g_k^{(r)}, g_k^{(i)}$ are

$$\begin{aligned}
g_0^{(r)} &= 1, & g_0^{(i)} &= 0,\\
g_1^{(r)} &= \tfrac{1}{2}\mu, & g_1^{(i)} &= 2\zeta,\\
g_2^{(r)} &= \tfrac{1}{24}(3\mu^2 + (4\pi^2 - 1)\mu - 48\zeta^2), & g_2^{(i)} &= (1+\mu)\zeta,\\
g_3^{(r)} &= \tfrac{1}{48}(\mu^3 + (4\pi^2 - 1)\mu^2 + 8(\pi^2 - 6\zeta^2)\mu - 96\zeta^2), & &\\
g_3^{(i)} &= \tfrac{1}{12}\zeta(3\mu^2 + (4\pi^2 + 5)\mu - 16\zeta^2 + 4). & &
\end{aligned} \tag{2.39}$$

The first-order approximation reads

$$B_n^\mu(z) = \frac{2\, n!\, n^{\mu-1}}{(2\pi)^n\, \Gamma(\mu)} \left[\cos\pi(2z + \mu - \tfrac{1}{2}n) + \mathcal{O}(1/n)\right], \quad n \to \infty. \tag{2.40}$$

Nörlund [6, p. 39] describes the same method of this section and only gives the first-order approximation.

2.3.1 An alternative expansion

As observed in the previous method, the main contributions to (2.1) comes from the singular points of the integrand at $\pm 2\pi i$. In this section we expand part of the integrand of (2.1) in a two-point Taylor expansion. In this way a simpler asymptotic representation can be obtained. For more details on this topic we refer to [2, 3] and for the evaluation of coefficients of such expansions to [13]. We write

$$f(w) = 2^{-3\mu}\pi^{-2\mu}\left(w^2 + 4\pi^2\right)^\mu \left(\frac{w}{e^w - 1}\right)^\mu e^{wz} \tag{2.41}$$

and expand

$$f(w) = \sum_{k=0}^{\infty} (\alpha_k + w\beta_k) \left(w^2 + 4\pi^2\right)^k . \tag{2.42}$$

The function $f(w)$ is analytic inside the disk $|w| < 4\pi$ and the series converges in the same domain. The coefficients α_0 and β_0 can be found by substituting $w = \pm 2\pi i$. This gives

$$\begin{aligned} \alpha_0 &= \frac{f(2\pi i) + f(-2\pi i)}{2} = \cos 2\zeta, \\ \beta_0 &= \frac{f(2\pi i) - f(-2\pi i)}{4\pi i} = \frac{1}{2\pi} \sin 2\zeta, \end{aligned} \tag{2.43}$$

where ζ is defined in (2.37).

The next coefficients can be obtained by writing $f_0(w) = f(w)$ and

$$\begin{aligned} f_{j+1}(w) &= \frac{f_j(w) - (\alpha_j + w\beta_j)}{w^2 + 4\pi^2} \\ &= \sum_{k=j+1}^{\infty} (\alpha_k + w\beta_k) \left(w^2 + 4\pi^2\right)^{k-j-1} , \end{aligned} \tag{2.44}$$

$j = 0, 1, 2, \ldots$, and by taking the limits when $w \to \pm 2\pi i$. We have

$$\begin{aligned} \alpha_{j+1} &= \frac{f_j'(2\pi i) - f_j'(-2\pi i)}{8\pi i}, \\ \beta_{j+1} &= -\frac{f_j'(2\pi i) + f_j'(-2\pi i) - 2\beta_j}{16\pi^2}. \end{aligned} \tag{2.45}$$

This gives

$$\begin{aligned} \alpha_1 &= -\frac{1}{16\pi^2}[3\mu \cos 2\zeta + 2\pi\eta \sin 2\zeta], \\ \beta_1 &= \frac{1}{32\pi^3}[2\pi\eta \cos 2\zeta + (2 - 3\mu) \sin 2\zeta], \\ \alpha_2 &= \frac{1}{1536\pi^4}[(-12\pi^2\eta^2 + 4\mu\pi^2 - 33\mu + 27\mu^2) \cos 2\zeta + 12\pi\eta(3\mu - 1) \sin 2\zeta], \\ \beta_2 &= \frac{1}{3072\pi^5}[-36\pi\eta(\mu - 1) \cos 2\zeta + (36 - 69\mu + 27\mu^2 + 4\mu\pi^2 - 12\pi^2\eta^2) \sin 2\zeta], \end{aligned} \tag{2.46}$$

where $\eta = \mu - 2z$.

Substituting the expansion in (2.42) into (2.1) we obtain

$$B_n^{\mu}(z) = n!\, 2^{3\mu} \pi^{2\mu} \sum_{k=0}^{\infty} \left[\alpha_k \Phi_k^{(n)} + \beta_k \Phi_k^{(n-1)}\right] , \tag{2.47}$$

where

$$\Phi_k^{(n)} = \frac{1}{2\pi i}\int_{\mathcal{C}} \left(w^2+4\pi^2\right)^{k-\mu} \frac{dw}{w^{n+1}}. \tag{2.48}$$

We have $\Phi_k^{(2n+1)} = 0$ and

$$\Phi_k^{(2n)} = (2\pi)^{2k-2\mu-2n} \binom{k-\mu}{n} = (-1)^n (2\pi)^{2k-2\mu-2n} \frac{(\mu-k)_n}{n!}. \tag{2.49}$$

Hence,

$$\begin{aligned} B_{2n}^{\mu}(z) &= (2n)!\, 2^{3\mu}\pi^{2\mu} \sum_{k=0}^{\infty} \alpha_k \Phi_k^{(2n)}, \\ B_{2n+1}^{\mu}(z) &= (2n+1)! 2^{3\mu}\pi^{2\mu} \sum_{k=0}^{\infty} \beta_k \Phi_k^{(2n)}. \end{aligned} \tag{2.50}$$

These convergent expansions have an asymptotic character for large n. This follows from (see (2.35))

$$\begin{aligned} \frac{\Phi_{k+1}^{(2n)}}{\Phi_k^{(2n)}} &= 4\pi^2 \frac{(\mu-k-1)_n}{(\mu-k)_n} \\ &= 4\pi^2 \frac{\Gamma(\mu-k-1+n)}{\Gamma(\mu-k-1)} \frac{\Gamma(\mu-k)}{\Gamma(\mu-k+n)} \\ &= 4\pi^2 \frac{\mu-k-1}{\mu-k+n-1} = \mathcal{O}\left(n^{-1}\right), \quad n\to\infty. \end{aligned} \tag{2.51}$$

We compare the first term approximations given in (2.40) and those from (2.50). From (2.40) we obtain

$$B_{2n}^{\mu}(z) \sim (-1)^n \frac{(2n)!\, 2^{\mu} n^{\mu-1}}{(2\pi)^{2n}\,\Gamma(\mu)} \cos\pi(2z+\mu) + \ldots, \tag{2.52}$$

and from (2.50)

$$B_{2n}^{\mu}(z) = (-1)^n \frac{(2n)!\, 2^{\mu}}{(2\pi)^{2n}\,\Gamma(\mu)} \frac{\Gamma(n+\mu)}{n!} \cos\pi(2z+\mu) + \ldots. \tag{2.53}$$

Because $\Gamma(n+\mu)/n! \sim n^{\mu-1}$ as $n\to\infty$, we see that the first approximations give the same asymptotic estimates, and they are exactly the same when $\mu = 1$.

Integer values of μ Comparing the expansions in (2.38) and (2.50), we observe that those in (2.50) do not vanish when $\mu = 0, -1, -2, \ldots$, whereas the expansion in (2.38) does. We have when $\mu = m$ (integer)

$$\Phi_k^{(2n)} = \begin{cases} (2\pi)^{2k-2m-2n} \binom{k-m}{n}, & k \geq n+m, \\ 0, & k < n+m. \end{cases} \tag{2.54}$$

Hence, the summation in (2.50) starts with $k = n + m$. The scale $\{\Phi_k^{2n}\}$ loses its asymptotic property, because now

$$\frac{\Phi_{k+1}^{(2n)}}{\Phi_k^{(2n)}} = 4\pi^2 \frac{n+\ell+1}{\ell+1} = \mathcal{O}(n), \quad n \to \infty, \tag{2.55}$$

where $k = n + m + \ell$, and a possible asymptotic character of the series in (2.50) has to be furnished by the coefficients α_k, β_k, which depend on n when $k \geq n + m$

Because n is assumed to be large, and the coefficients α_k, β_k in (2.50) become quite complicated when $k \geq n + m$, these expansions are of no use when μ is an integer.

When we replace the expansion in (2.42) with

$$f(w) = \sum_{k=0}^{\infty} \left(\widetilde{\alpha}_k + w\widetilde{\beta}_k\right) \left(\frac{w^2+4\pi^2}{w^2}\right)^k \tag{2.56}$$

we obtain

$$\begin{aligned} B_{2n}^{\mu}(z) &\sim (2n)!\, 2^{3\mu} \pi^{2\mu} \sum_{k=0}^{\infty} \widetilde{\alpha}_k \widetilde{\Phi}_k^{(2n)}, \\ B_{2n+1}^{\mu}(z) &\sim (2n+1)! 2^{3\mu} \pi^{2\mu} \sum_{k=0}^{\infty} \widetilde{\beta}_k \widetilde{\Phi}_k^{(2n)}, \end{aligned} \tag{2.57}$$

where $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ can be obtained from a similar scheme as in (2.44)-(2.45). The functions $\Phi_k^{(2n)}$ are given by

$$\widetilde{\Phi}_k^{(2n)} = \frac{(-1)^{n+k}}{(2\pi)^{2\mu+2n}} \frac{(\mu-k)_{n+k}}{(n+k)!} = \frac{(-1)^{n+k}}{(2\pi)^{2\mu+2n}} \frac{\Gamma(\mu+n)}{\Gamma(\mu-k)\,(n+k)!}. \tag{2.58}$$

When $\mu = m$ (integer) these functions vanish if $k - m = 0, 1, 2, \ldots$, which is more useful that in the earlier choice (2.42). When $m < 0$ all terms vanish, when $m > 0$ the series have a finite number of terms.

With the expansion in (2.56), which converges in certain neighborhoods of the points $w = \pm 2\pi i$ (and not in a domain that contains any allowed deformation of the curve $\mathcal{C}$ in (2.1)), the expansions in (2.57) do not converge, but they have an asymptotic character for large n.

As an example, when $m = 1$ the expansions in (2.57) have just one term ($k = 0$). In this case $\widetilde{\Phi}_0^{(2n)} = (-1)^n/(2\pi)^{2n+2}$ and $\widetilde{\alpha}_0 = \alpha_0$, $\widetilde{\beta}_0 = \beta_0$ (see (2.43)). These approximations correspond exactly to the first terms in the expansions in (2.10) and (2.11).

3 The generalized Euler polynomials

We can use the same methods as for the Bernoulli polynomials, and, therefore, we give less details. Again, three different cases arise, depending on $\mu = 0, -1, -2, \ldots$, $\mu = 1, 2, 3, \ldots$, and μ otherwise, real or complex. In the first and third case we use the Cauchy integral

$$E_n^\mu(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{2^\mu e^{wz}}{(e^w + 1)^\mu} \frac{dw}{w^{n+1}}, \tag{3.1}$$

where $\mathcal{C}$ is a circle around the origin, with radius less than π. This follows from (1.2).

3.1 Asymptotic form when $\mu = 0, -1, -2, \ldots$

We proceed as in §2.1 and write $\mu = -m$ ($m = 0, 1, 2, \ldots$). We expand $E_n^\mu(z)$ in terms of a finite sum. We have

$$E_n^{-m}(z) = 2^{-m} \sum_{r=0}^{m} \binom{m}{r} (z + r)^n. \tag{3.2}$$

For any given $m \in \mathbb{N}$ and complex z only the term or terms with the largest values of $|z + r|$ will give a large contribution to the sum in (3.2), the other terms being exponentially small in comparison. We conclude, that (3.2) gives the asymptotic form when $n \to \infty$, when m and z are fixed. In particular, when $z > 0$, the term with index $r = m$ is maximal, and we have

$$E_n^{-m}(z) = 2^{-m}(z + m)^n \left[1 + \mathcal{O}\left(\frac{z + m - 1}{z + m}\right)^n\right]. \tag{3.3}$$

The error term can also be estimated by $\mathcal{O}(\exp(-n/(z + m)))$, which is indeed exponentially small compared with unity.

For general complex $z = x + iy$ and $x > -m/2$ the term with index $r = m$ again is maximal and the same estimate as in (3.3) is valid. When $x = -m/2$ the terms with $r = 0$ and $r = m$ give the maximal contributions, and we have

$$E_n^{-m}(z) \sim 2^{-m} \left[(-\tfrac{1}{2}m + iy)^n + (\tfrac{1}{2}m + iy)^n \right]. \tag{3.4}$$

When $x < -m/2$ the term with index $r = 0$ is maximal, and we have

$$E_n^{-m}(z) = 2^{-m} z^n \left[1 + \mathcal{O}\left(\frac{z+1}{z} \right)^n \right]. \tag{3.5}$$

As explained at the end of §2.1 these estimates can also be derived by using the saddle point method.

3.2 Asymptotic form when $\mu = 1, 2, 3, \ldots$

We write $\mu = m$. For $m = 1$ we have

$$\begin{aligned} E_n^1(z) = E_n(z) &= 2\,n! \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)\pi i z}}{((2k+1)\pi i)^{n+1}} \\ &= 4\,n! \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi z - \frac{1}{2}\pi n)}{((2k+1)\pi)^{n+1}}, \end{aligned} \tag{3.6}$$

where $z \in (0, 1)$ if $n = 0$ and $z \in [0, 1]$ if $n > 0$. This expansion follows from (3.1) as in §2.2.

In the second series in (3.6) only the term with $k = 0$ is relevant for the asymptotic behavior, and we obtain for fixed complex z

$$E_n^1(z) = \frac{4\,n!}{\pi^{n+1}} \left[\sin\left(\pi z - \tfrac{1}{2}\pi n \right) + \mathcal{O}\left(3^{-n}\right) \right], \quad n \to \infty. \tag{3.7}$$

For general fixed real or complex z the series in (3.6) can be viewed as asymptotic expansion for large n, as easily follows from the ratio test.

For general $\mu = m = 1, 2, 3, \ldots$ a similar expansion as in (3.6) can be given. In that case the poles of the integrand in (3.1) are of higher order. We can write

$$E_n^m(z) = 2\,n! \sum_{k=0}^{\infty} \epsilon_k^m(n, z) \frac{e^{(2k+1)\pi i z}}{((2k+1)\pi i)^{n+1}}, \tag{3.8}$$

where $\epsilon_k^1(n, z) = 1, \forall k$.

To obtain $\epsilon_k^m(n,z)$ for $m>1$ we compute the residues of the poles at $(2k+1)\pi i$ of order m of the integrand in (3.1). For this we compute the coefficient d_{m-1} in the expansion

$$\frac{(w-(2k+1)\pi i)^m e^{zw}}{(e^w+1)^m w^{n+1}} = \sum_{r=0}^{\infty} d_r (w-(2k+1)\pi i)^r. \tag{3.9}$$

We substitute $w = s + (2k+1)\pi i$ and write the expansion as

$$(-1)^m e^{z(2k+1)\pi i} \frac{s^m e^{zs}}{(e^s-1)^m (s+(2k+1)\pi i)^{n+1}} = \sum_{r=0}^{\infty} d_r s^r. \tag{3.10}$$

We use (1.1) and conclude that

$$d_{m-1} = \frac{(-1)^m e^{z(2k+1)\pi i}}{((2k+1)\pi i)^{n+1}} \sum_{\nu=0}^{m-1} \frac{B_\nu^m(z)}{\nu!} \binom{-n-1}{m-1-\nu} ((2k+1)\pi i)^{\nu+1-m}. \tag{3.11}$$

It follows that

$$\epsilon_k^m(n,z) = (-1)^{m-1} 2^{m-1} \sum_{\nu=0}^{m-1} \frac{B_\nu^m(z)}{\nu!} \binom{-n-1}{m-1-\nu} ((2k+1)\pi i)^{\nu+1-m}, \tag{3.12}$$

which we write in the form

$$\begin{aligned}\epsilon_k^m(n,z) = {} & \frac{2^{m-1}}{((2k+1)\pi i)^{m-1}} \binom{n+m-1}{m-1} \times \\ & \sum_{\nu=0}^{m-1} B_\nu^m(z) \binom{m-1}{\nu} \frac{(n+m-\nu-1)!}{(n+m-1)!} (-(2k+1)\pi i)^\nu.\end{aligned} \tag{3.13}$$

For large n the main term occurs for $\nu = 0$, giving

$$\epsilon_k^m(n,z) = \frac{2^{m-1} n^{m-1}}{(m-1)!\,((2k+1)\pi i)^{m-1}} \left[1 + \mathcal{O}\left(n^{-1}\right)\right], \tag{3.14}$$

and in (3.8) the terms with $k = 0, -1$ give the main terms, and we obtain for fixed m and complex z (cf. (3.7))

$$\begin{aligned}E_n^m(z) = {} & \frac{2^{m+1} n!}{\pi^{n+m}} \binom{n+m-1}{m-1} \times \\ & \left[\sum_{\nu=0}^{m-1} B_\nu^m(z) \binom{m-1}{\nu} \frac{(n+m-\nu-1)!}{(n+m-1)!} \pi^\nu \sin\tau + \mathcal{O}\left(3^{-n}\right)\right],\end{aligned} \tag{3.15}$$

as $n \to \infty$, where $\tau = (z - \frac{1}{2}n - \frac{1}{2}(m-1) - \frac{1}{2}\nu)\pi$.

3.3 Asymptotic form for general complex $\boldsymbol{\mu}$

The analysis is as in §2.3. We use a contour for the integral (3.1) as in Figure 1, now with loops around the branch points $\pm\pi i$, and with radius of the large circle smaller than 3π. We denote the integrals around the loops by $I_\pm$. After the substitution $w = \pi i \exp(s)$ we obtain for the upper loop

$$I_+ = \frac{2^\mu n!}{2\pi i}\frac{e^{\pi i z-\mu\pi i}}{(\pi i)^{n+\mu}}\int_{\mathcal{C}_+} h(s)s^{-\mu}e^{-ns}\,ds, \tag{3.16}$$

where

$$h(s) = e^{zu}\left(\frac{\pi i s}{e^u-1}\right)^\mu, \quad u = \pi i\,(e^s-1)\,. \tag{3.17}$$

We expand $h(s) = \sum_{k=0}^\infty h_k s^k$ and interchange summation and integration in (3.16). By using (2.33) and (2.34) we obtain the result

$$E_n^\mu(z) \sim \frac{2^{\mu+1}\,n!\,n^{\mu-1}}{\pi^{n+\mu}\,\Gamma(\mu)}\left[\cos\chi\sum_{k=0}^\infty\frac{(1-\mu)_k h_k^{(r)}}{n^k} - \sin\chi\sum_{k=0}^\infty\frac{(1-\mu)_k h_k^{(i)}}{n^k}\right], \tag{3.18}$$

as $n\to\infty$, with z and μ fixed complex numbers ($\mu\notin\mathbb{Z}$), where

$$\chi = \zeta - \tfrac{1}{2}n\pi, \quad \zeta = (z-\tfrac{1}{2}\mu)\pi. \tag{3.19}$$

The first few coefficients $h_k^{(r)}, h_k^{(i)}$ are

$$\begin{aligned}
h_0^{(r)} &= 1, & h_0^{(i)} &= 0,\\
h_1^{(r)} &= -\tfrac{1}{2}\mu, & h_1^{(i)} &= \zeta,\\
h_2^{(r)} &= \tfrac{1}{24}(3(1-2\pi^2)\mu^2+(13\pi^2-12\zeta\pi-1)\mu-12\zeta^2), & h_2^{(i)} &= \tfrac{1}{2}(1-\mu)\zeta,\\
h_3^{(r)} &= \tfrac{1}{48}z(-\mu^3+(1-\pi^2)\mu^2+2(\pi^2+6\zeta^2)\mu-24\zeta^2), &&\\
h_3^{(i)} &= \tfrac{1}{24}\zeta(3\mu^2+(\pi^2-7)\mu-4\zeta^2+4). &&
\end{aligned} \tag{3.20}$$

The first-order approximation reads

$$E_n^\mu(z) = \frac{2^{\mu+1}\,n!\,n^{\mu-1}}{\pi^{n+\mu}\,\Gamma(\mu)}\left[\cos\pi(z-\tfrac{1}{2}\mu-\tfrac{1}{2}n)+\mathcal{O}(1/n)\right], \quad n\to\infty. \tag{3.21}$$

3.3.1 An alternative expansion

We repeat the steps of §2.3.1. We write

$$g(w) = \left(\frac{w^2+\pi^2}{2\pi}\right)^{\mu} \left(\frac{1}{e^w+1}\right)^{\mu} e^{wz} \tag{3.22}$$

and expand

$$g(w) = \sum_{k=0}^{\infty} (\gamma_k + w\delta_k)\left(w^2+\pi^2\right)^k. \tag{3.23}$$

We have

$$\begin{aligned}\gamma_0 &= \frac{g(\pi i)+g(-\pi i)}{2} = \cos\zeta,\\ \delta_0 &= \frac{g(\pi i)-g(-\pi i)}{2\pi i} = \frac{1}{\pi}\sin\zeta.\end{aligned} \tag{3.24}$$

where $\zeta = (z - \frac{1}{2}\mu)\pi$.

The next coefficients follow from writing $g_0(w) = g(w)$ and

$$\begin{aligned} g_{j+1}(w) &= \frac{g_j(w) - (\gamma_j + w\delta_j)}{w^2+\pi^2} \\ &= \sum_{k=j+1}^{\infty} (\gamma_k + w\delta_k)\left(w^2+\pi^2\right)^{k-j-1}, \quad j = 0,1,2,\ldots. \end{aligned} \tag{3.25}$$

This gives

$$\begin{aligned}\gamma_{j+1} &= \frac{g_j'(\pi i) - g_j'(-\pi i)}{4\pi i},\\ \delta_{j+1} &= -\frac{g_j'(\pi i) + g_j'(-\pi i) - 2\delta_j}{4\pi^2}.\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\gamma_1 &= -\frac{1}{4\pi^2}[\mu\cos\zeta + \pi\eta\sin\zeta],\\ \delta_1 &= \frac{1}{4\pi^3}[\pi\eta\cos\zeta + (2-\mu)\sin\zeta],\\ \gamma_2 &= \frac{1}{96\pi^4}[(-9\mu - 3\pi^2\eta^2 + \pi^2\mu + 3\mu^2)\cos\zeta + 6\pi\eta(\mu-1)\sin\zeta],\\ \delta_2 &= \frac{1}{96\pi^5}[6\pi\eta(3-\mu)\cos\zeta + (36 - 21\mu + 3\mu^2 + \pi^2\mu - 3\pi^2\eta^2)\sin\zeta],\end{aligned} \tag{3.27}$$

where $\eta = \mu - 2z$.

Substituting the expansion in (3.23) into (3.1) we obtain

$$E_n^{\mu}(z) = (4\pi)^{\mu} n! \sum_{k=0}^{\infty} \left[\gamma_k \Psi_k^{(n)} + \delta_k \Psi_k^{(n-1)}\right], \tag{3.28}$$

where

$$\Psi_k^{(n)} = \frac{1}{2\pi i}\int_{\mathcal{C}} \left(w^2+\pi^2\right)^{k-\mu} \frac{dw}{w^{n+1}}. \tag{3.29}$$

We have $\Psi_k^{(2n+1)} = 0$ and

$$\Psi_k^{(2n)} = \pi^{2k-2\mu-2n}\binom{k-\mu}{n} = (-1)^n\pi^{2k-2\mu-2n}\frac{(\mu-k)_n}{n!}. \tag{3.30}$$

Hence,

$$\begin{aligned} E_{2n}^{\mu}(z) &= (2\pi)^{\mu}(2n)!\sum_{k=0}^{\infty}\gamma_k\Psi_k^{(2n)},\\ E_{2n+1}^{\mu}(z) &= (2\pi)^{\mu}(2n+1)!\sum_{k=0}^{\infty}\delta_k\Psi_k^{(2n)}. \end{aligned} \tag{3.31}$$

These convergent expansions have an asymptotic character for large n. This follows from

$$\frac{\Psi_{k+1}^{(2n)}}{\Psi_k^{(2n)}} = \pi^2\frac{\mu-k}{\mu-k-1+n} = \mathcal{O}\left(n^{-1}\right), \quad n\to\infty. \tag{3.32}$$

Comparing the first term approximations given in (3.21) and those from (3.31) we obtain from (3.21)

$$E_{2n}^{\mu}(z) \sim (-1)^n\frac{(2n)!\,2^{2\mu}n^{\mu-1}}{\pi^{2n+\mu}\,\Gamma(\mu)}\cos\pi(z-\tfrac12\mu)+\ldots, \tag{3.33}$$

and from (3.31)

$$E_{2n}^{\mu}(z) = (-1)^n\frac{(2n)!\,2^{2\mu}}{\pi^{2n+\mu}\,\Gamma(\mu)}\frac{\Gamma(n+\mu)}{n!}\cos\pi(z-\tfrac12\mu)+\ldots. \tag{3.34}$$

and we see that the first approximations give the same asymptotic estimates.

Integer values of $\boldsymbol{\mu}$ The expansions in (3.31) do not vanish when μ is a negative integer, as the expansion in (3.18) does. We have when $\mu = m$ (integer)

$$\Psi_k^{(2n)} = \begin{cases} \pi^{2k-2m-2n}\binom{k-m}{n}, & k\ge n+m,\\ 0, & k< n+m. \end{cases} \tag{3.35}$$

Hence, the summation in (3.31) starts with $k = n+m$.

When we expand

$$g(w) = \sum_{k=0}^{\infty} \left(\widetilde{\gamma}_k + w\widetilde{\delta}_k\right) \left(\frac{w^2+\pi^2}{w^2}\right)^k \tag{3.36}$$

we obtain the expansions

$$\begin{aligned} E_{2n}^{\mu}(z) &\sim (2\pi)^{\mu}(2n)! \sum_{k=0}^{\infty} \widetilde{\gamma}_k \widetilde{\Psi}_k^{(2n)}, \\ E_{2n+1}^{\mu}(z) &\sim (2\pi)^{\mu}(2n+1)! \sum_{k=0}^{\infty} \widetilde{\delta}_k \widetilde{\Psi}_k^{(2n)}, \end{aligned} \tag{3.37}$$

where $\widetilde{\gamma}_k$ and $\widetilde{\delta}_k$ can be obtained from a similar scheme as in (3.26). The functions $\widetilde{\Psi}_k^{(2n)}$ are given by

$$\widetilde{\Psi}_k^{(2n)} = \pi^{-2\mu-2n} \binom{k-\mu}{n+k} = (-1)^{n+k} \pi^{-2\mu-2n} \frac{(\mu-k)_{n+k}}{(n+k)!}. \tag{3.38}$$

When $\mu = m$ (integer) these functions vanish if $k - m = 0, 1, 2, \ldots$, which is more useful that in the earlier choice (3.23). For example, when $m = 1$ the expansions in (3.37) have just one term ($k = 0$). In this case $\widetilde{\Psi}_0^{(2n)} = (-1)^n/\pi^{2n+2}$ and $\widetilde{\gamma}_0 = \gamma_0$, $\widetilde{\delta}_0 = \delta_0$ (see (3.24)). These approximations correspond exactly to the first term in the second expansion in (3.6).

Acknowledgments. The authors thank the referee for the constructive remarks. The *Gobierno of Navarra, Res. 07/05/2008* is acknowledged for its financial support. JLL acknowledges financial support from *Ministerio de Educación y Ciencia*, project MTM2007–63772. NMT acknowledges financial support from *Ministerio de Educación y Ciencia*, project MTM2006–09050.

References

[1] J. L. López and N. M. Temme. Hermite polynomials in asymptotic representations of generalized Bernoulli, Euler, Bessel, and Buchholz polynomials. *J. Math. Anal. Appl.*, 239(2):457–477, 1999.

[2] José L. López and Nico M. Temme. Two-point Taylor expansions of analytic functions. *Stud. Appl. Math.*, 109(4):297–311, 2002.

[3] José L. López and Nico M. Temme. Multi-point Taylor expansions of analytic functions. *Trans. Amer. Math. Soc.*, 356(11):4323–4342 (electronic), 2004.

[4] Albert T. Lundell. On the denominator of generalized Bernoulli numbers. *J. Number Theory*, 26(1):79–88, 1987.

[5] L. M. Milne-Thomson. *The Calculus of Finite Differences*. Macmillan and Co., Ltd., London, 1951.

[6] N. E. Nörlund. Sur les valeurs asymptotiques des nombres et des polynômes de Bernoulli. *Rend. Circ. Mat. Palermo (2)*, 10:27–44, 1961.

[7] N.E. Nörlund. *Vorlesungen über Differenzenrechnung*. Springer, Berlin, 1924.

[8] F. W. J. Olver. *Asymptotics and Special Functions*. AKP Classics. A K Peters Ltd., Wellesley, MA, 1997. Reprint, with corrections, of original Academic Press edition, 1974.

[9] H. M. Srivastava and Pavel G. Todorov. An explicit formula for the generalized Bernoulli polynomials. *J. Math. Anal. Appl.*, 130(2):509–513, 1988.

[10] N. M. Temme. *Special Functions*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1996. An introduction to the classical functions of mathematical physics.

[11] Pavel G. Todorov. Une formule simple explicite des nombres de Bernoulli généralisés. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(13):665–666, 1985.

[12] Pavel G. Todorov. Explicit and recurrence formulas for generalized Euler numbers. *Funct. Approx. Comment. Math.*, 22:13–17 (1994), 1993.

[13] Raimundas Vidunas and Nico M. Temme. Symbolic evaluation of coefficients in Airy-type asymptotic expansions. *J. Math. Anal. Appl.*, 269(1):317–331, 2002.

[14] A. Weinmann. Asymptotic expansions of generalized Bernoulli polynomials. *Proc. Cambridge Philos. Soc.*, 59:73–80, 1963.